

Hamiltonian, Energy and Entropy in General Relativity with Non-Orthogonal Boundaries

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Abstract

A general recipe to define, via Nöther theorem, the Hamiltonian in any natural field theory is suggested. It is based on a Regge–Teitelboim-like approach applied to the variation of Nöther conserved quantities. The Hamiltonian for General Relativity in presence of non-orthogonal boundaries is analysed and the energy is defined as the on-shell value of the Hamiltonian. The role played by boundary conditions in the formalism is outlined and the quasilocal internal energy is defined by imposing metric Dirichlet boundary conditions. A (conditioned) agreement with previous definitions is proved. A correspondence with Brown-York original formulation of the first principle of black hole thermodynamics is finally established.

1 Introduction

One of the most popular approaches to black hole thermodynamics is due to Brown and York; see [1, 2]. They defined, through an Hamilton–Jacobi analysis of a gravitational system, the quasilocal energy, the quasilocal angular momentum and the spatial stress of a spatially bounded region of spacetime. Those quasilocal surface densities were subsequently involved in describing the first law of black hole thermodynamics through a statistical analysis based on path integral methods. As a result, in the final expression of the first law, quasilocal physical quantities are involved rather than the global ones calculated asymptotically. Some advantages may, in fact, be gained by considering spatially bounded self-gravitating systems instead of considering the properties of the whole universe. From a mathematical viewpoint, if we analyze only a spatially bounded region, the asymptotic behavior of the gravitational field becomes irrelevant and different gravitational solutions with different asymptotic behaviors can be handled on equal footing. On the other hand, from a physical viewpoint it was noticed that a system in thermal equilibrium must feature a finite spatial

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extent: a system of infinite spatial extent at fixed temperature is thermodynamically unstable (see e.g. [3]). Hence, one could argue that, in establishing the first principle of black hole thermodynamics, the role of internal energy should be played by a suitable defined “surface energy” rather than by the total mass (calculated in asymptotic regions). Correspondingly, if the temperature is identified with the intensive variable which is thermodynamically conjugated to the energy, it has to depend somehow on the surface rather than to be identified with a constant parameter. The same reasoning can be applied to all the relevant physical quantities which are expected to appear in the first law of black hole thermodynamics. This is the idea at the basis of the aforementioned papers [1], [2].

Nevertheless the formalism developed in those papers deals only with foliations of spacetime with spatial boundaries orthogonal to the timelike boundary.

In [4, 5] the assumption of orthogonal boundaries was removed and the quasilocal formalism of Brown–York was extended in order to deal with the general case. The formalism there developed starts from the so-called Trace-K gravitational action functional as introduced by Hayward in [6]. Roughly speaking, the Trace-K action differs from the standard Hilbert action by metric dependent boundary and corner terms and it features a variational principle with fixed boundary metric. The canonical analysis of [4, 5] goes on through an ADM splitting of the Trace-K action functional. The Hamiltonian ensues from the canonical reduction of the action; the energy (or it would be better to say the mass) is defined as the on-shell value of the Hamiltonian and turns out to be a pure boundary term. By focusing just on the (space + time) foliation of the timelike boundary, the formalism turns out to be independent on the foliation of spacetime inside or outside the boundary. The expression of the quasilocal energy for foliations with non-orthogonal boundaries was applied in [4] to study the Lorentz-like transformation rules of the quasilocal energy with respect to boosted observers. In particular, the energy measured by observers infalling towards a black hole was calculated. Moreover, the close analogy between the quasilocal formalism and the thin shell formalism of Israel (see [7]) was analysed in [4] with the aim of physically supporting and motivating the interpretation of energy attributed to the mathematical formula for energy there suggested.

A similar formulation was also given by Hawking and Hunter in [8]. However the analysis there developed was carried on by considering the foliation of the spacetime as a whole rather than focusing just on the foliation of the boundary. The Hawking–Hunter definition of the gravitational Hamiltonian features a “tilting term” which has the disadvantage to depend on the intersection angle of the boundaries. In order to remove this unacceptable term a careful choice of the background has to be imposed; see [8].

Another definition of energy for non-orthogonal boundaries can be also found in Kijowski’s paper [9]. Such a definition was there obtained by means of a symplectic analysis of the gravitational Lagrangian and through boundary Legendre transformations. The main contribution of Kijowski’s approach is related to the way in which different boundary conditions are related to different definitions of energy (i.e. internal energy or free energy). Starting from the same gravitational

Lagrangian, namely the Hilbert Lagrangian, different definitions of energy are recovered in [9] through different control modes of the boundary data. Moreover, different boundary conditions reverberate in different Legendre transformations or, in other words, in different boundary symplectic structures.

Many attempts to describe the quasilocal stress–energy content of a spatially bounded region of spacetime via Nöther theorem can be also found elsewhere; see, e.g. [10, 11, 12, 13] and references quoted therein. Nevertheless it was long but wrongly believed – see [5] – that a Nöther framework for conserved quantities is not well–suited to deal with foliations of spacetime with non–orthogonal boundaries. The purpose of this paper is to contradict this common belief. Indeed, we shall show that a Regge–Teitelboim–like approach to Nöther theorem allows to handle boundary terms in such a way that quasilocal Nöther charges can be defined. Roughly speaking, the leading idea is to define *the variation of conserved quantities* δQ along a 1–parameter family of field configurations rather than defining the conserved quantities Q in their own. It is in fact well–known that *absolute* conserved quantities do not have a proper physical meaning while it is instead meaningful to define *relative* conserved quantities, i.e. conserved quantities with respect to a fixed solution chosen as a background and which is introduced with the aim to define the “zero level” for conserved quantities themselves; see [10, 14, 15, 16, 17]. The background solution has to be chosen carefully (and differently case by case) because it has to satisfy appropriate boundary matching conditions. However it is also well–known that the problem of finding a suitable background, if possible, can be rather cumbersome in direct applications. Nevertheless, at a theoretical level, the problem can be bypassed by defining the variation δQ as suggested above. We remind that to each infinitesimal Lagrangian symmetry ξ one can associate via Nöther theorem a conserved quantity $Q(\xi)$ which is made of a bulk term $Q_{\text{bulk}}(\xi)$ plus a surface term $Q_{\text{surface}}(\xi)$. On a spacetime M of arbitrary dimension m ($m \neq 2$) the bulk term is obtained by integrating on a $(m - 1)$ dimensional region Σ a suitable $(m - 1)$ –form $\tilde{\mathcal{E}}(\xi)$ which is called the *reduced Nöther current* and which is vanishing on shell, i.e. by setting $Q_{\text{bulk}}(\xi) = \int_{\Sigma} \tilde{\mathcal{E}}(\xi)$. The surface term $Q_{\text{surface}}(\xi)$ is instead obtained by integrating on the boundary $\partial\Sigma$ the $(m - 2)$ –form $\mathcal{U}(\xi)$ which is called the *superpotential* and satisfies a canonical splitting $\mathcal{E} = \tilde{\mathcal{E}} + d\mathcal{U}$, where \mathcal{E} is the generator of Nöther currents (see e.g. [18, 19]), i.e. by setting $Q_{\text{surface}}(\xi) = \int_{\partial\Sigma} \mathcal{U}(\xi)$. The superpotential is algorithmically calculated for every natural or gauge–natural theory starting from the Lagrangian of the theory and the vector field ξ ; see [14, 18]. If we consider a Cauchy hypersurface Σ in spacetime and a vector field ξ transverse to Σ the Hamiltonian can be a priori defined for a Lagrangian theory as the Nöther charge relative to ξ , i.e. $H(\xi) = Q(\xi)$. Namely, the Hamiltonian is the Nöther charge associated to the “time” displacement vector field ξ , having identified the (local) flow parameter of ξ with the (local) time. Nevertheless this definition does not produce the expected values when specialized to specific solutions (it is indeed affected by anomalous factors or divergence problems) mainly because of the fact that such a definition does not take into proper account a reference background. Therefore, this prescription leads to a definition of conserved quantities which has to

be somehow corrected. To this end one defines the variation $\delta\hat{Q}$ of the *corrected* conserved quantity \hat{Q} by adding to δQ a suitable additional boundary term. The latter ensues from the variation δQ_{bulk} of the bulk term and it keeps into account the behavior of the fields at the boundary. In this way the formalism resembles the original idea of Regge–Teitelboim (see [20]), i.e. boundary terms arising in the variation of the bulk Hamiltonian are added (with a minus sign) to the very definition of Hamiltonian so that the variation of the new Hamiltonian does not contain boundary terms at all. However, once the variation $\delta\hat{Q}$ of the corrected conserved quantity is defined, it remains to analyse the problem if the variation $\delta\hat{Q}$ can be formally integrated. We shall see that this problem is tightly related with the boundary conditions we choose (i.e. it is related to the control boundary data). When $\delta\hat{Q}$ is integrable we obtain the conserved quantity $\hat{Q} - \hat{Q}_0$ defined up to a constant of integration \hat{Q}_0 . The latter can be fixed as a zero level for the conserved quantity or, in other words, as a background reference. Namely, \hat{Q}_0 corresponds to the Nöther charge relative to a solution inside the set of solutions satisfying the same boundary (or asymptotic) conditions (and, in turn, it fixes the boundary or asymptotic conditions themselves). Notice, however, that the suitable matching conditions between a solution and its background are required only on the boundary (or at infinity) so that the background is allowed to have a different topology (for example, this is the case if we fix the Minkowski background for the Schwarzschild solution; see [13] for details).

The variation of Nöther conserved quantities was already considered in [21]. This approach was at the basis of the so-called *covariant ADM formalism*. The formalism has been deeply analysed and tested in its applications to General Relativity and it was shown that it reproduces the expected physical values of conserved quantities. Nevertheless, as far as we know, all the examples so far analysed in General Relativity cover only the non-orthogonal boundary case. It is our goal in this paper to generalise and extend the aforementioned formalism in order to fill this gap, also in view of a unifying perspective on the various formalisms recalled above and sometimes apparently unrelated. To this end a new additional surface term has to be added to the Nöther-based definition of Hamiltonian when dealing with non orthogonal boundaries. Indeed the variation of the corrected Hamiltonian $\hat{H}(\xi) = \hat{Q}(\xi)$ gives rise to an additional boundary term which is instead vanishing in presence of orthogonal intersections. Again, this term has to be added to \hat{H} with a minus sign so that, eventually, the variation of the Hamiltonian does not really contain surface terms. It is also remarkable that the final formula we shall obtain for the Hamiltonian is independent on divergence terms added to the Lagrangian, so that it depends only on the field equations content of the equivalence class $[L]$ of all such dynamically equivalent Lagrangians. From a mathematical viewpoint this is clearly an advantage. Indeed, we do not have to care about surface terms in the action functional. On the contrary, other Hamiltonian definitions based on a framework different from the Nöther one (such as the ones based on the canonical reduction of the action functional or based on a Hamilton–Jacobi analysis) are deeply sensitive to boundary terms. In those formulations, in fact,

boundary terms reflect into boundary conditions which have to be satisfied by the dynamical fields. Different surface terms in the action functional then correspond to different boundary conditions and, in the end, to different definitions of energy. Indeed, it is important to notice that for most physical systems there may exist different kinds of energy, each one corresponding to different choices of boundary conditions and/or of control variables. For instance, if we agree that the internal energy of a system is the energy corresponding to Dirichlet boundary conditions, when we want to calculate the internal energy for a solution of Lagrangian field equations, we have to go back to the Lagrangian itself and to look for divergence terms which have to be added to the Lagrangian to obtain the appropriate Dirichlet boundary conditions (in General Relativity the sought-for Lagrangian corresponds to the Hayward Lagrangian [6]). Moreover, when we try to consider a self-gravitating system as a thermodynamical system, different boundary conditions reverberate on different choices between the fields which have to be considered as intensive or extensive thermodynamical variables; see [2]. Accordingly to the choice made on thermodynamical variables (i.e. on the boundary control modes of the system) one has then to exhibit different action functionals, e.g. the microcanonical action functional, the canonical partition function or the grand canonical partition function. In this way the initial Lagrangian must be modified each time according to the model one is dealing with.

On the other hand, the definition of the Hamiltonian based on Nöther theorem is unique inside each class $[L]$ of Lagrangians, where the elements of each class $[L]$ differ from each other only by total divergences¹. No *ad hoc* physical or mathematical prescription is then required to select a well-defined representative inside $[L]$ because all of them lead to the same Hamiltonian. Nevertheless, if the Hamiltonian is unique for any element of the class, the question can arise how different definitions of energy can be recovered in this framework. In other words, how different boundary conditions can be imposed on the same Hamiltonian? (A similar problem was analysed by Kijowski in [9] and also in [10, 15]). In our case the answer has to be looked for into the way we consider the variation of the Hamiltonian. Namely, we can consider different one-parameter families of solutions each one containing the same solution (i.e. in the infinite dimensional space of solutions we consider different curves passing through the same point). Different one-parameter families of solutions (i.e. different curves) correspond to different ways in which the variation of the Hamiltonian is carried out and, in turn, to different definitions of energy.

The plan of the paper is as follows. In Section 2 we shall present the theoretical framework leading to the definition of Hamiltonian and energy in relativistic field theories. We stress that even though the formalism will be presented with the aim of being applicated to General Relativity, it will be developed inside a geometrical framework which is common to all natural as well as to all gauge-natural relativistic theories so that, at least in principle, all classical theories of physical interest can be covered. Starting from Section 3 we shall focus only on

¹ This is a cohomology class in the proper variational cohomology theory; see, e.g. [22, 23].

General Relativity which for simplicity will be considered only in vacuum. The generalization to cases with matter is an easy task which adds nothing to the understanding of the problem and, as such, we leave it to some interested reader. It is in fact the second order Hilbert Lagrangian (as well as any Lagrangian of order $k > 1$) which is sensitive to the formalism presented here; first order matter Lagrangians can in fact be treated in the usual way; see [14, 21]. First of all we shall shortly review the geometry of the non-orthogonal foliation of (a portion of) spacetime and we shall fix the notation. Then, in subsection 3.1, the variation of the Hamiltonian together with Hamilton equations are calculated in order to test the formalism. The quasilocal energy of the gravitational field, defined as the on-shell value of the Hamiltonian, is analysed in subsection 3.2. It will be there shown that the variation of the energy can be formally integrated provided Dirichlet boundary conditions are imposed. In this way a definition of quasilocal *internal* energy for a spatially bounded gravitating system is obtained whence the reference background subtraction terms are properly taken into account. Moreover, a comparison with the alternative definitions of internal energy given in [4, 8, 9] is carried out and a (conditioned) correspondence is provided. Finally, in subsection 3.3, the first principle of black-hole thermodynamics is easily obtained as a trivial consequence of the cohomological properties of Nöther charges.

2 Nöther Theorem

We shall here consider a field theory described by a Lagrangian of order k . According to the geometric approach to field theory (see for instance [19, 24, 25, 26]) we are assuming that the configuration bundle of the theory is a bundle (Y, M, π) . The base manifold M , with $\dim M = m$, is the space of parameters of the theory. From now we shall identify the manifold M with the physical spacetime. Fibered adapted coordinates on Y are (x^μ, y^i) , where $\mu = 1, \dots, m = \dim M$, $i = 1, \dots, n = \dim Y - \dim M$. A *configuration* is a (local) section $\sigma : M \longrightarrow Y$, locally $x^\lambda \mapsto (x^\lambda, y^i = \sigma^i(x^\mu))$, which to each point $x \in M$ associates the values of the fields “ y ” (of components y^i). Thence, a configuration completely describes the evolution of the fields which represent a given physical model. The fibered manifold $(J^k Y, \pi^k, M)$ denotes the k jet prolongation of the configuration bundle (i.e. the bundle where fields live together with their spacetime derivatives up to order k included) and $j^k \sigma : M \longrightarrow J^k Y$ denotes the jet prolongation of the section σ (i.e. j^k denotes derivation up to order k). Natural fibered coordinates on $J^k Y$ are obviously defined and denoted by $(x^\mu, y^i, \dots, y_{\mu_1 \dots \mu_k}^i)$. In the framework so far developed a Lagrangian of order k can be seen as an horizontal form (or equivalently as a base preserving fibered morphism) $L : J^k Y \longrightarrow \Lambda^m(M)$. It can be locally written as $L = \mathcal{L}(y^i, \dots, y_{\mu_1 \dots \mu_k}^i) ds$ where \mathcal{L} is a scalar density, called the *Lagrangian density*, and $ds = dx^1 \wedge \dots \wedge dx^m$ is the local volume element of m -dimensional spacetime. Once the Lagrangian is evaluated on (the jet prolongation of) a section it gives rise to a m -form $(j^k \sigma)^* L$ on spacetime M .

A vertical vector field X is a section of the vertical bundle $VY \longrightarrow Y$, i.e. it is a vector field on the configuration bundle Y which is everywhere tangent to the fibers (or, equivalently, projects down to zero under the tangent map of the bundle projection). Locally, it reads as $X = X^i \partial / \partial y^i = (\delta y^i) \partial / \partial y^i$ and it describes what is classically called the “variation” δy^i of the dynamical fields. The k -jet prolongation $J^k X$ is a section of the bundle $J^k(VY) \longrightarrow Y$ and, roughly speaking, it describes the variation $\delta y^i, \delta y_\mu^i, \dots, \delta y_{\mu_1 \dots \mu_k}^i$ of the dynamical fields together with their derivatives up to order k .

Given a vertical vector field X and a section σ and denoting by ϕ_τ the (local) flow of X we define a one-parameter family σ_τ of sections through the rule $\sigma_\tau = \phi_\tau \circ \sigma$. The variation $\delta_X \omega$ with respect to X of a differential form ω on any $J^h Y$ ($h \geq 1$) is defined according to the rule

$$\delta_X \omega = \left. \frac{d}{d\tau} \omega(j^h \sigma_\tau) \right|_{\tau=0}$$

From now on, for notational simplicity, we shall assume that the Lagrangian, as well as all horizontal forms on any prolongation of the configuration bundle, are already computed along sections so they actually define differential forms on spacetime, e.g. L stands shortly for $(j^k \sigma)^* L$. In other words we use the shortcut y^i to actually denote $y^i = \sigma^i(x)$, i.e. the i -th component of the section σ with respect to the i -th fibered coordinate y^i . Analogously y_μ^i is used instead of $\partial_\mu \sigma^i(x)$ and so on for the derivatives $j^h y$ of higher order. Nevertheless we emphasize that all calculations we shall carry on in the standard language of differential calculus on the base manifold M can be alternatively carried on at the level of jet bundle calculus and only at the end the results so obtained can be pulled back on spacetime. In the latter case globality and uniqueness problems are easily solved and calculations can be algorithmically defined. Since we are here interested in applications, we shall skip the rigorous geometric framework and we refer the reader to [14, 18] for a deeper insight into the mathematical details of the matter. To stimulate the interest of physically-oriented readers we shall keep as much as possible the formal setting to a minimum and we shall use a coordinate language.

Given a vertical vector field X , the variation $\delta_X \mathcal{L}$ of the Lagrangian $L = \mathcal{L}(j^k y) ds$ can be generally written, through a well known integration by parts procedure, as follows:

$$\delta_X \mathcal{L}(j^k y) = e_i(j^{2k} y) X^i + d_\lambda F^\lambda(j^{2k-1} y, j^{k-1} X) \quad (1)$$

or, in terms of differential forms, as

$$\delta_X L = e(L, X) + dF(L, X) \quad (2)$$

Here the m -form

$$e(L, X) = e_i(j^{2k} y) X^i ds \quad (3)$$

and the $(m-1)$ -form

$$F(L, X) = F^\lambda(j^{2k-1} y, j^{k-1} X) ds_\lambda, \quad ds_\lambda = i_\lambda ds \quad (4)$$

are called, respectively, *the Euler–Lagrange form* and *the Poincaré–Cartan form*. We stress that the Poincaré–Cartan form $F(L, X)$ depends linearly on X together with its derivatives up to the order $k - 1$ included. [We also remark that, for higher order theories the Poincaré–Cartan form is not unique since it usually depends on the choice of a linear connection on spacetime; see [27]. We shall not care about this in the following since we shall only deal with General Relativity where a canonical Poincaré–Cartan form can be chosen.]

The *critical sections* $\sigma : x^\lambda \mapsto y^i = \sigma^i(x)$ are those which satisfy the Euler–Lagrange equations:

$$e_i(j^{2k}y)|_{y=\sigma(x)} = 0 \quad (5)$$

We shall now consider *natural theories*, namely, theories describing the behavior of geometric objects (such, e.g., tensor fields, tensor densities or linear connections) by means of Lagrangians which are covariant with respect to the action of diffeomorphisms of spacetime M . In mathematical language natural theories are the ones fulfilling the *fundamental identity*

$$d(i_\xi L) = \left\{ \frac{\partial \mathcal{L}}{\partial y^i} \mathcal{L}_\xi y^i + \dots + \frac{\partial \mathcal{L}}{\partial y_{\mu_1 \dots \mu_k}^i} \mathcal{L}_\xi y_{\mu_1 \dots \mu_k}^i \right\} ds \quad (6)$$

for each vector field ξ on spacetime; see [19]. Inserting the first variation formula (2) (with X replaced by $\mathcal{L}_\xi y$) into the right hand side of the fundamental identity (6) we obtain the conservation laws

$$d\mathcal{E}(L, \xi) = \mathcal{W}(L, \xi) \quad (7)$$

where the $(m - 1)$ -form $\mathcal{E}(L, \xi)$ and the m -form $\mathcal{W}(L, \xi)$ are defined, respectively, as follows:

$$\mathcal{E}(L, \xi) = F(L, \mathcal{L}_\xi y) - i_\xi L \quad (8)$$

$$\mathcal{W}(L, \xi) = -e(L, \mathcal{L}_\xi y) \quad (9)$$

and are called, respectively, *the Nöther current* of L relative to ξ and *the work current*. Notice that, by definition, one has $\mathcal{W}(L, \xi)|_{y=\sigma(x)} = 0$ whenever σ is a solution of field equations, so that $\mathcal{W}(L, \xi)$ evaluates by (7) the “work” performed off-shell, i.e. by sections which are not critical. Accordingly, from equation (7) we infer that the differential form $\mathcal{E}(L, \xi)$ is conserved (i.e. closed) on-shell. Moreover, in all natural theories the map $\xi \mapsto \mathcal{E}(L, \xi)$ is a linear partial differential operator in the coefficients ξ^μ . Hence, the Nöther current $\mathcal{E}(L, \xi)$, through a (covariant) integration by parts, can be rewritten as

$$\mathcal{E}(L, \xi) = \tilde{\mathcal{E}}(L, \xi) + d\mathcal{U}(L, \xi) \quad (10)$$

where the $(m - 1)$ -form $\tilde{\mathcal{E}}(L, \xi)$ is called *the reduced current* and it is vanishing on-shell because it is proportional to a combination of field equations; see [18, 14]. The $(m - 2)$ -form $\mathcal{U}(L, \xi)$ is instead called *the superpotential*. It depends on the fields y and their derivatives up to order $2k - 2$ and it is linear in the

components ξ^μ and their derivatives up to order $k - 2$. Hence, it follows from (10) that the Nöther current \mathcal{E} is not only closed but it is also exact on-shell.² Since $\mathcal{E}(L, X)$ is an $(m - 1)$ -form in spacetime, it can be integrated over an $(m - 1)$ -dimensional region Σ , namely, a submanifold Σ of M with a boundary $\partial\Sigma \subset \Sigma \subset M$ which, in turn, is a compact $(m - 2)$ -dimensional submanifold. With this formalism at hands, one could be tempted to define the Nöther charge $Q_\Sigma(L, \xi, \sigma)$ along a section σ and relative to ξ as follows:

$$\begin{aligned} Q_\Sigma(L, \xi, \sigma) &= \int_\Sigma \mathcal{E}(L, \xi)|_{y=\sigma(x)} \\ &= \int_\Sigma \tilde{\mathcal{E}}(L, \xi)|_{y=\sigma(x)} + \int_{\partial\Sigma} \mathcal{U}(L, \xi)|_{y=\sigma(x)} \end{aligned} \quad (11)$$

(notice that the Nöther charge $Q_\Sigma(L, \xi, \sigma)$ becomes a pure boundary integral whenever σ is a solution of field equations). Nevertheless it is well known that the definition given in this way does not fit the physical expectation values.

Example 2.1 The Hilbert Lagrangian

$$L_H = \frac{1}{2\kappa} \sqrt{g} g^{\mu\nu} R_{\mu\nu} ds \quad (12)$$

(with $\kappa = 8\pi$ in geometric units with $c = G = 1$) is a second order Lagrangian ($k = 2$) on the configuration bundle $Lor(M) \rightarrow M$ of Lorentzian metric on the 4 dimensional spacetime M . A vertical vector field on the configuration bundle reads locally as $X = (\delta g_{\mu\nu}) \partial / \partial g_{\mu\nu}$. The variation of the Lagrangian can be written as:

$$\delta L_H = -\frac{1}{2\kappa} \sqrt{g} G^{\mu\nu} \delta g_{\mu\nu} ds + d \left[\frac{\sqrt{g}}{2\kappa} g^{\mu\nu} \delta u_{\mu\nu}^\alpha ds_\alpha \right] \quad (13)$$

where

$$u_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - \delta_{(\mu}^\alpha \Gamma_{\nu)\sigma}^\sigma \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad (14)$$

Hence, the Eule–Lagrange form and the Poincaré–Cartan form are given, respectively, by

$$e(L_H, X) = -\frac{1}{2\kappa} \sqrt{g} G^{\mu\nu} \delta g_{\mu\nu} ds \quad (15)$$

$$F(L_H, X) = \frac{\sqrt{g}}{2\kappa} g^{\mu\nu} \delta u_{\mu\nu}^\alpha ds_\alpha \quad (16)$$

The Nöther current, defined as

$$\mathcal{E}(L_H, \xi) = \frac{1}{2\kappa} \{ g^{\mu\nu} \mathcal{L}_\xi(u_{\mu\nu}^\alpha) - \xi^\alpha L_H \} ds_\alpha$$

²We stress that the Nöther current as well as the superpotential are *algorithmically* and *uniquely* defined at the bundle level in terms of jet bundle morphisms and they are *canonically* associated to the Lagrangian. Then they are computed along a configuration σ and they give rise to differential forms on spacetime.

splits as $\mathcal{E}(L_H, \xi) = \tilde{\mathcal{E}}(L_H, \xi) + d\mathcal{U}(L_H, \xi)$ where

$$\tilde{\mathcal{E}}(L_H, \xi) = \frac{\sqrt{g}}{\kappa} G_\nu^\alpha \xi^\nu ds_\alpha \quad (17)$$

$$\mathcal{U}(L_H, \xi) = \frac{\sqrt{g}}{2\kappa} g^{\nu\rho} \nabla_\rho \xi^\mu ds_{\mu\nu} \quad (18)$$

Notice that the superpotential $\mathcal{U}(L_H, \xi)$ is nothing but the Komar superpotential; see [28]. When integrated on spacelike hypersurfaces, it is well-known that the Komar superpotential does not reproduce the expected values for all conserved quantities. Indeed it is affected by the so-called “anomalous factor problem” when computed along asymptotically flat solutions (see [29]) or, even worse, it gives rise to divergence problems on solutions of Einstein’s equations not featuring an asymptotically flat behavior (see [30, 31]).

This example strongly suggests that a modification in the definition of Nöther charges has to be provided. \blacksquare

Let us consider again a general natural theory. According to definition (8), for each vertical vector field $X = (\delta y^i) \partial/\partial y^i$ we have:

$$\begin{aligned} \delta_X \mathcal{E}(L, \xi) &= \delta_X F(L, \mathcal{L}_\xi y) - i_\xi (\delta_X L) \stackrel{(2)}{=} \delta_X F(L, \mathcal{L}_\xi y) - i_\xi e(L, X) - i_\xi dF(L, X) \\ &= \delta_X F(L, \mathcal{L}_\xi y) - \mathcal{L}_\xi F(L, X) - i_\xi e(L, X) + di_\xi F(L, X) \\ &= \omega(L, X, \mathcal{L}_\xi y) - i_\xi e(L, X) + d(i_\xi F(L, X)) \end{aligned} \quad (19)$$

where *the symplectic current* $\omega(L, X, Y)$ relative to L and calculated for two vertical vector fields X and Y is defined as (see [32]):

$$\omega(L, X, Y) = \delta_X F(L, Y) - \delta_Y F(L, X) \quad (20)$$

Equation (19) suggests to redefine the variation $\delta_X \hat{Q}$ of the corrected conserved quantity \hat{Q} as follows

$$\begin{aligned} \delta_X \hat{Q}_\Sigma(L, \xi) &= \int_\Sigma [\delta_X \mathcal{E}(L, \xi) - d(i_\xi F(L, X))] \\ &\stackrel{(10)}{=} \int_\Sigma \delta_X \tilde{\mathcal{E}}(L, \xi) + \int_{\partial\Sigma} [\delta_X \mathcal{U}(L, \xi) - i_\xi F(L, X)] \end{aligned} \quad (21)$$

Hence, from equation (19) we obtain:

$$\delta_X \hat{Q}_\Sigma(L, \xi) = \int_\Sigma \{\omega(L, X, \mathcal{L}_\xi y) - i_\xi e(L, X)\} \quad (22)$$

This prescription is at the basis of the so-called ADM *covariant* formalism; see [14, 21], as well as it is at the basis of a Hamiltonian analysis through a symplectic viewpoint (see [11, 12, 32]). Indeed, whenever the (nowhere vanishing) vector field ξ is transverse to a Cauchy hypersurface Σ we define the (variation of the) Hamiltonian $\hat{H}(L, \xi, \Sigma)$ simply as follows:

$$\delta_X \hat{H}(L, \xi, \Sigma) = \delta_X \hat{Q}_\Sigma(L, \xi) \quad (23)$$

i. e. we identify the variation of the Hamiltonian with the variation of the *corrected* Nöther charge \hat{Q}_Σ relative to ξ . First of all notice that definition (23) has the advantage of being independent on divergence terms added to the Lagrangian. Indeed if we consider a new Lagrangian L' differing from L by a total divergence, i.e. $L' = L + d\beta$, where β is a $(m-1)$ -form, we have

$$\begin{aligned}\tilde{\mathcal{E}}(L', \xi) &= \tilde{\mathcal{E}}(L, \xi) \\ \mathcal{U}(L', \xi) &= \mathcal{U}(L, \xi) + i_\xi \beta \\ i_\xi F(L', X) &= i_\xi F(L, X) + i_\xi \delta_X \beta\end{aligned}$$

and hence $\delta_X \hat{Q}_\Sigma(L', \xi) = \delta_X \hat{Q}_\Sigma(L, \xi)$; see definition (21). This means that divergence terms added to the Lagrangian just affect the superpotential so that this fact would reverberate on the definition of the Hamiltonian if it were naively defined through the non corrected Nöther charge (11). Namely, different divergence terms would imply different definitions of Hamiltonian and, in turn, different “boundary Hamilton equations”. Instead, the boundary term $i_\xi F(L, X)$ added in the modified definition (21) exactly counterbalances the modifications in the superpotential and we end up with a unique definition of Hamiltonian (unique inside the set of Lagrangians differing from each other only by divergence terms). This property is clearly welcome. The duty of the Hamiltonian should in fact be to dictate, via Hamilton equations, the evolution laws of the dynamical fields in such a way that Hamilton equations have to correspond to (the space+time decomposition of) Euler–Lagrange equations. As well as these latter equations are insensitive to Lagrangian divergence terms one would expect the same property to hold true also for Hamilton equations.

Notice also that, according to definitions (21) and (23), the (variation of the) Hamiltonian consists of a bulk term plus a surface term. It is exactly the surface term which plays a key role in testing the viability in the definition of the Hamiltonian. Indeed the variation of the Hamiltonian is well-defined if all surface terms in δH vanish. That is, the surface terms in the Hamiltonian, when varied, should exactly cancel out surface terms arising in the variation of the bulk term. In this way, the variation of the Hamiltonian would be well-defined on the full phase space of the dynamical fields rather than just on the restricted space of fields satisfying suitable boundary conditions (namely, the boundary conditions necessary to cancel out possible surface terms in δH). Hence, in order to give a viable definition of Hamiltonian, we propose here an additional boundary modification to the general prescription (21). We stress that, when dealing with General Relativity, the following modification will turn out to be relevant only when the foliation of (a region of) spacetime does not feature orthogonal intersections with the timelike boundary.

Let us thence consider a (portion of a) Cauchy hypersurface Σ in spacetime. Since the Poincaré–Cartan form $F(L, X)$ is linear in X and its derivatives, owing to definition (20) the integral $\int_\Sigma \omega(L, X, \mathcal{L}_\xi y)$ of the symplectic form on the hypersurface Σ , by means of an integration by parts procedure, can be

rewritten as follows:

$$\int_{\Sigma} \omega(L, X, \mathcal{L}_{\xi} y) = \int_{\Sigma} \tilde{\omega}(L, X, \mathcal{L}_{\xi} y) + \int_{\partial\Sigma} \tau(L, X, \mathcal{L}_{\xi} y) \quad (24)$$

i.e. it splits into a bulk term plus a pure boundary term which, for $k > 1$, is in general non vanishing (and we shall see below that this is indeed the case for General Relativity). Accordingly, our proposal is to slightly modify definition (21) by pushing the boundary term $\int_{\partial\Sigma} \tau$ directly into the definition of the Hamiltonian and defining then a new surface-adapted Hamiltonian $H(L, \xi, \Sigma)$. That is, we define the variation $\delta_X H$ of the new Hamiltonian as follows

$$\begin{aligned} \delta_X H(L, \xi, \Sigma) &= \int_{\Sigma} \{ \delta_X \mathcal{E}(L, \xi) - d[i_{\xi} F(L, X) + \tau(L, X, \mathcal{L}_{\xi} y)] \} \\ &= \int_{\Sigma} \delta_X \tilde{\mathcal{E}}(L, \xi) + \int_{\partial\Sigma} [\delta_X \mathcal{U}(L, \xi) - i_{\xi} F(L, X) - \tau(L, X, \mathcal{L}_{\xi} y)] \end{aligned} \quad (25)$$

This prescription is very close in spirit to the original idea of Regge–Teitelboim (see [20]) even though we are now dealing with any natural field theory and the prescription is based on a symplectic analysis of Nöther theorem instead of being based on a canonical analysis of the ADM Hamiltonian. It follows in fact from (22) and (24) that:

$$\delta_X H(L, \xi, \Sigma) = \int_{\Sigma} \tilde{\omega}(L, X, \mathcal{L}_{\xi} y) - \int_{\Sigma} i_{\xi} e(L, X) \quad (26)$$

and equation (26) ensures that $\delta_X H$ does not contain surface integrals which, as already explained, would correspond to Hamiltonian boundary conditions on the dynamical fields.

Notice also that the definition (25) does not alter the property of invariance of δH with respect to the addition of divergence terms into the Lagrangian. Nevertheless the definition given depends in general on the particular hypersurface Σ we are dealing with. Additional boundary hypotheses have then to be imposed in order to avoid the problem. We refer the reader to [32] where the dependence of the symplectic current on the Cauchy hypersurface was carefully investigated. We shall here skip the details on the matter and we shall instead focus on the definition of energy which is our main concern in this paper.

We define the variation of the *energy* $E(L, \xi)$ relative to ξ as the on shell value of the variation of the Hamiltonian. Hence, the variation $\delta_X E(L, \xi)$ is obtained evaluating (25) along solutions of field equations and considering variations with respect to a vertical vector field X which, in turn, is a solution of linearized field equations, which in our formalism are equivalent to the equations

$$\delta_X \tilde{\mathcal{E}}(L, \xi) = 0 \quad (27)$$

Since under these hypotheses we have $i_{\xi} e(L, X) = 0$, from definition (25) we obtain that $\delta_X E(L, \xi)$ turns out to be a pure boundary term, i.e.

$$\delta_X E(L, \xi) \doteq \int_{\partial\Sigma} \delta_X \mathcal{U}(L, \xi) - i_{\xi} F(L, X) - \tau(L, X, \mathcal{L}_{\xi} y) \quad (28)$$

while equation (26) reduces to

$$\delta_X E(L, \xi) = \int_{\Sigma} \tilde{\omega}(L, X, \mathcal{L}_{\xi} y) \quad (29)$$

Notice however that the expressions (25) and (28) define only the variation of the Hamiltonian and of the energy, respectively. The problem arises whether those expressions can be formally integrated or not. We shall see that, at least for General Relativity, this problem admits a positive answer. Roughly speaking, the problem of integrability is related to the way in which we keep the field variations or, in other words, is related to the way we impose boundary conditions (e.g. Dirichlet or Neumann conditions). Notice that in the present framework for boundary conditions we actually mean conditions which have to be satisfied by the vertical vector fields X on the boundary $\partial\Sigma$ (for example $X|_{\partial\Sigma} = 0$). Different classes of vector fields correspond to different sets of boundary conditions and they will give rise to different definitions of energy, as it is physically expected. Notice that boundary conditions are not imposed, a priori, by adding divergences terms to the Lagrangian we started from. As already outlined in the Introduction, this would be a rather unpleasant strategy mainly because we would have to modify the theory from the very beginning, i.e. at the Lagrangian level, accordingly on what kind of energy we are interested in (e.g. internal energy, free energy and so on). Indeed we stress again that each kind of energy is deeply related to its own boundary conditions (see [9, 15, 10]) and in the Hamiltonian formulations based on the canonical reduction of the action functional, boundary conditions are dictated by divergence terms added to the Lagrangian and vice versa. Hence, in order to satisfy different boundary conditions we would have to add different divergence terms each time. In other words, each symplectic control–response boundary structure dictates its own action functional.

On the contrary our Nöther–based prescription is insensitive on divergence terms so that definition (28) is unchanged if we remain inside any class $[L]$ of Lagrangians, the elements of which consist of Lagrangians differing from each other only by divergence terms. Different physical meanings can be nevertheless attributed to the formula (28), which is the same for each element of the class $[L]$, by considering variations with respect to different classes of vertical vector fields X . Namely, boundary conditions play an effective role only at the Hamiltonian level. The vector field X (together with its jet prolongation) keeps track of what has to be held fixed on the boundary, i.e. it determines the control variables. In this way the Lagrangian formulation gets rid of a direct physical meaning which is instead restored at the Hamiltonian level. Indeed, it is only in the Hamiltonian framework where we read off the symplectic structure of the theory, namely the control–response variables and the corresponding energetic content; see [9, 15]. We can completely ignore divergence terms and the corresponding boundary conditions in the Lagrangian formulation provided we assign to the Lagrangian the only duty to frame the theory in a well–defined geometric framework and to provide the equations of motion³. Instead, at the Hamiltonian level, we are

³ Notice that, in the geometric approach to Lagrangian theories, the equations of motion are

free to fix the boundary conditions in any way we want through the choice of appropriate vector fields.

The application to General Relativity will help us to clarify in detail how the idea works on.

3 General Relativity

As an application of the formalism let us now consider again standard General Relativity in vacuum with $\dim M = 4$ (notice however that the formalism goes through for each spacetime dimension $m > 2$. The case $m = 2$ will be instead treated separately in a forthcoming paper [34]).

Let us consider a 4-dimensional region $D \subseteq M$ which is diffeomorphic to the product $\Sigma \times \mathbb{R}$ where Σ is a 3-dimensional closed manifold with boundary $B = \partial\Sigma$. We denote this diffeomorphism by

$$\psi : \Sigma \times \mathbb{R} \longrightarrow D \quad (30)$$

For simplicity we assume in the present Section that the boundary $B = \partial\Sigma$ is topologically a sphere (even though no relevant change arises when we endow Σ with an internal boundary, i.e. when the manifold Σ is assumed to be the product $S^2 \times I$ where I is a closed real interval). For any $t \in \mathbb{R}$ a hypersurface $\Sigma_t \subset D$ is induced by ψ according to the rule $\Sigma_t = \psi(\Sigma \times t) = \psi_t(\Sigma)$ and we require it to be a spacelike Cauchy hypersurface. Each map $\psi_t : \Sigma \longrightarrow \Sigma_t$ establishes an embedding of Σ into M . The set of all Σ_t , for t in \mathbb{R} , defines a foliation of D labelled by the time parameter t . Moreover, each Σ_t intersects the boundary ∂D in a compact 2 dimensional surface B_t which is diffeomorphic to B , for all t in \mathbb{R} ; see Figure 1. The diffeomorphism is established by the map $\psi_t : B \longrightarrow B_t$. Hence the boundary $\mathcal{B} = \partial D$ is a timelike hypersurface globally diffeomorphic to the product manifold $B \times \mathbb{R}$. The time evolution field ξ in D is defined through the (local) rule $\xi^\mu \nabla_\mu t = 1$ and, on the boundary \mathcal{B} , it is tangent to the boundary itself. We shall use Greek letters, ranging from 0 to 3, to label indices of tensorial objects on D . Tensors on Σ are labelled by Latin letters a, b, \dots running from 1 to 3. Tensors on $B \times \mathbb{R}$ are labelled by middle Latin letters $i, j, \dots = 0, 2, 3$; while Capital latin letters $A, B, \dots = 2, 3$ are used for tensors on B . Notice that tensorial objects $T_{\alpha\beta\dots}^{\mu\nu\dots}$ on M , with contravariant indices $\mu, \nu \dots$ tangent to each Σ_t , can be pulled back on Σ via the embedding maps ψ_t and they originate tensor fields $T_{ab\dots}^{cd\dots}$ on Σ . Accordingly these tensor fields have a double interpretation. When labelled by Greek indices they are thought of as tensor fields in spacetime M , whilst when they are labelled by Latin indices they are instead thought of as tensor on Σ . The same double notation applies, with the corresponding modifications, for tensor fields which

mathematically recovered from the variation $\delta_X A_D(\sigma)$ of the action functional by postulating that critical sections σ are the ones extremising the action for *any* region D and for *any* vector field X with compact support in D . Hence all boundary integrals in $\delta_X A_D(\sigma)$ are vanishing due to the hypothesis made on X . In this way action functionals differing from each other by boundary terms give rise to the same equations of motion.

are “tangent” to \mathcal{B} and tensor fields on $B \times \mathbb{R}$; and also for tensor fields “tangent” to B_t and tensor fields on B (where “tangent” is referred to each contravariant index).

We denote by u^μ the future directed unit normal to Σ_t and we denote by n_μ the outward pointing unit normal of B_t in Σ_t . Of course $u^\mu n_\mu|_{B_t} = 0$. Notice that we are not requiring that the hypersurfaces Σ_t intersect the timelike boundary \mathcal{B} orthogonally so that, in general, one has $\xi^\mu n_\mu|_{B_t} \neq 0$. We also denote by \bar{n}^μ the outward pointing unit normal of \mathcal{B} in M and with \bar{u}^μ the future directed unit normal of B_t in \mathcal{B} . We have $\xi^\mu \bar{n}_\mu|_{B_t} = \bar{u}^\mu \bar{n}_\mu|_{B_t} = 0$. The evolution vector field can be decomposed as $\xi^\mu = N u^\mu + N^\mu$ where the shift vector N^μ is tangent to the hypersurfaces Σ_t , i.e. $N^\mu u_\mu = 0$ (but in general $N^\mu n_\mu|_{B_t} \neq 0$). The metrics induced on Σ_t , \mathcal{B} and B_t by the metric $g_{\mu\nu}$ are given, respectively, by:

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \quad (31)$$

$$\bar{\gamma}_{\mu\nu} = g_{\mu\nu} - \bar{n}_\mu \bar{n}_\nu \quad (32)$$

$$\sigma_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu = \bar{\gamma}_{\mu\nu} + \bar{u}_\mu \bar{u}_\nu \quad (33)$$

(Notice that h_{ab} , $\bar{\gamma}_{ij}$ and σ_{AB} will instead denote the pull back metrics on Σ , $B \times \mathbb{R}$ and B , respectively.) We shall denote by h , $\bar{\gamma}$ and σ , respectively, the absolute values of the metric determinants. The metrics (31), (32) and (33) with an index raised through the contravariant metric $g^{\mu\nu}$ define the projection operators in the corresponding surfaces. We also denote by

$$K_{\mu\nu} = -h_\mu^\alpha \nabla_\alpha u_\nu \quad (34)$$

$$\bar{\Theta}_{\mu\nu} = -\bar{\gamma}_\mu^\alpha \nabla_\alpha \bar{n}_\nu \quad (35)$$

$$\mathcal{K}_{\mu\nu} = -\sigma_\mu^\alpha D_\alpha n_\nu \quad (36)$$

respectively, the extrinsic curvatures of Σ_t in M , of \mathcal{B} in M and of B_t in Σ_t . The symbols D and \bar{D} denote, respectively, the (metric) covariant derivative on Σ_t compatible with $h_{\mu\nu}$ and the covariant derivative on \mathcal{B} compatible with the metric $\bar{\gamma}_{\mu\nu}$. We shall denote by $K = K_{\mu\nu} h^{\mu\nu}$, $\bar{\Theta} = \bar{\Theta}_{\mu\nu} \bar{\gamma}^{\mu\nu}$ and $\mathcal{K} = \mathcal{K}_{\mu\nu} \sigma^{\mu\nu}$ the traces of the appropriate extrinsic curvatures. The momentum $P^{\mu\nu}$ of the hypersurface Σ_t is defined as:

$$P^{\mu\nu} = \frac{\sqrt{h}}{2\kappa} (K h^{\mu\nu} - K^{\mu\nu}) \quad (37)$$

while for the momentum $\bar{\Pi}^{\mu\nu}$ of the hypersurface \mathcal{B} we set:

$$\bar{\Pi}^{\mu\nu} = -\frac{\sqrt{\bar{\gamma}}}{2\kappa} (\bar{\Theta} \bar{\gamma}^{\mu\nu} - \bar{\Theta}^{\mu\nu}) \quad (38)$$

From a physical viewpoint the surface B_t can be thought of as an instantaneous configuration of a set of observers at the time t and their history is represented by the timelike boundary \mathcal{B} ; see [4]. The *unboosted*, or unbarred, observers are the ones evolving with four velocity u^μ and hence orthogonally to the foliation

Σ_t . They are at rest with respect to the Σ foliation. They view u^μ and n^μ as the unit normals to B_t . We instead refer to the *boosted*, or barred, observers to the ones which are on the same surface B_t at the instant of time t but evolving with velocity ξ^μ . They are comoving with \mathcal{B} . Since $\xi^\mu \in T\mathcal{B}$ then boosted observers view \bar{u}^μ and \bar{n}^μ as the unit normal to B_t . Barred and unbarred observers are related by the boost relations:

$$\bar{u}_\mu = \gamma u_\mu + \gamma v n_\mu \quad (39)$$

$$\bar{n}_\mu = \gamma n_\mu + \gamma v u_\mu \quad (40)$$

where

$$\gamma v = \bar{u}_\mu n^\mu = -u_\mu \bar{n}^\mu = \sinh \theta \quad (41)$$

measures the non-orthogonality of the intersections of the leaves Σ_t with the timelike boundary \mathcal{B} . The parameter θ is usually called the *velocity parameter*. Moreover we set $\gamma = (1 - v^2)^{-1/2}$ where

$$v = -\frac{\xi^\mu n_\mu}{\xi^\mu u_\mu} = \frac{N^\mu n_\mu}{N} \quad (42)$$

is the boost velocity between the observers. It measures the velocity in the direction of n^μ of an object with four velocity ξ^μ as measured by the observer with velocity u^μ , namely the velocity in the n^μ direction of boosted observers with respect to unboosted observers.

The vector field ξ^μ , as already said, can be decomposed as $\xi^\mu = N u^\mu + N^\mu$. On the boundary \mathcal{B} it can be also decomposed as $\xi^\mu = \bar{N} \bar{u}^\mu + \bar{N}^\mu$ where $\bar{N} = N/\gamma$ and $\bar{N}^\mu = \sigma_\nu^\mu N^\nu$. From now on we shall use the notation that a bar over a quantity will denote that it is defined with respect to the normals \bar{u}^μ and \bar{n}^μ , so that, e.g., $\bar{\mathcal{K}}_{\mu\nu}$ would correspond to the extrinsic curvature of B_t embedded in a spacelike hypersurface with unit normal \bar{u}^μ in B_t (or, roughly speaking, it is the extrinsic curvature of B_t with respect to the normal \bar{n}^μ). Moreover, the following identities hold true:

$$\sqrt{g} = N \sqrt{h}, \quad \sqrt{\bar{g}} = \bar{N} \sqrt{\bar{\sigma}} \quad (43)$$

3.1 Hamiltonian

Let us now consider the Hilbert Lagrangian L_H . According to the general prescription (25) the variation of the Hamiltonian is defined as follows:

$$\begin{aligned} \delta_X H(L_H, \xi, \Sigma_t) &= \int_{\Sigma_t} \delta_X \tilde{\mathcal{E}}(L_H, \xi) \\ &+ \int_{B_t} [\delta_X \mathcal{U}_K(L_H, \xi) - i_\xi F(L_H, X) - \tau(L_H, X, \mathcal{L}_\xi g)] \end{aligned} \quad (44)$$

where (see example 2.1) we set:

$$F(L_H, X) = \frac{\sqrt{g}}{2\kappa} g^{\mu\nu} \delta u_{\mu\nu}^\alpha ds_\alpha \quad (45)$$

$$\tilde{\mathcal{E}}(L_H, \xi) = \frac{\sqrt{g}}{\kappa} G_\nu^\alpha \xi^\nu ds_\alpha = \{N\mathcal{H} + N^\alpha \mathcal{H}_\alpha\} d^3x \quad (46)$$

$$\mathcal{U}_K(L_H, \xi) = \frac{\sqrt{g}}{2\kappa} g^{\nu\rho} \nabla_\rho \xi^\mu ds_{\mu\nu} \quad (47)$$

and \mathcal{H} and \mathcal{H}_α in (46) are called, respectively, the Hamiltonian constraint and the diffeomorphism constraints.

Now, our next task will be to calculate the term $\tau(L_H, X, \mathcal{L}_\xi g)$ and to rewrite (44) in terms of quantities adapted to the spacetime foliation. In this way it will be possible to establish whether formula (44) is a viable definition for the Hamiltonian. Moreover a correspondence with other definitions existing in literature will be possible. From now on we shall adopt the notational choices made in [5] and we shall make repeated use of useful formulae there demonstrated. Let us start with the relations

$$g^{\alpha\beta} \delta u_{\alpha\beta}^\mu u_\mu = 2\delta K + K^{\mu\nu} \delta h_{\mu\nu} + D_\mu (h_\alpha^\mu \delta u^\alpha) \quad (48)$$

$$g^{\alpha\beta} \delta u_{\alpha\beta}^\mu \bar{n}_\mu = 2\delta \bar{\Theta} + \bar{\Theta}^{\mu\nu} \delta \bar{\gamma}_{\mu\nu} + \bar{\mathcal{D}}_\mu (\bar{\gamma}_\alpha^\mu \delta \bar{n}^\alpha) \quad (49)$$

(the proof of which can be found in the aforementioned paper [5]). From (45), (48) and Stokes' theorem it then follows that

$$\begin{aligned} \int_{\Sigma_t} F^\mu(L_H, X) ds_\mu &= \frac{1}{2\kappa} \int_{\Sigma_t} \sqrt{g} g^{\alpha\beta} \delta u_{\alpha\beta}^\mu ds_\mu = -\frac{1}{2\kappa} \int_{\Sigma_t} g^{\alpha\beta} \delta u_{\alpha\beta}^\mu u_\mu \sqrt{h} d^3x \\ &= -\frac{1}{2\kappa} \int_{\Sigma_t} \{2\delta K + K^{\mu\nu} \delta h_{\mu\nu}\} \sqrt{h} d^3x - \frac{1}{2\kappa} \int_{B_t} n_\alpha h_\mu^\alpha \delta u^\mu \sqrt{\sigma} d^2x \\ &= -\int_{\Sigma_t} h_{\mu\nu} \delta P^{\mu\nu} d^3x - \frac{1}{2\kappa} \int_{B_t} n_\mu \delta u^\mu \sqrt{\sigma} d^2x \end{aligned} \quad (50)$$

(notice that $\int_{\Sigma_t} \sqrt{g} f^\mu ds_\mu = -\int_{\Sigma_t} f^\mu u_\mu \sqrt{h} d^3x$ for any 3-form $f = f^\mu ds_\mu$ and $\int_{\Sigma_t} D_\mu X^\mu \sqrt{h} d^3x = \int_{B_t} X^\mu n_\mu \sqrt{\sigma} d^2x$ for a vector field X^μ). Let us now consider the integral of the symplectic current (20) on the generic leaf Σ_t of the foliation. From (50) we obtain:

$$\int_{\Sigma_t} \omega(L_H, X, \mathcal{L}_\xi g) = \int_{\Sigma_t} \tilde{\omega}(L_H, X, \mathcal{L}_\xi g) + \int_{B_t} \tau(L_H, X, \mathcal{L}_\xi g) \quad (51)$$

where

$$\tilde{\omega}(L_H, X, \mathcal{L}_\xi g) = \{(\mathcal{L}_\xi h_{\mu\nu}) \delta P^{\mu\nu} - (\mathcal{L}_\xi P^{\mu\nu}) \delta h_{\mu\nu}\} d^3x \quad (52)$$

$$\tau(L_H, X, \mathcal{L}_\xi g) = \frac{1}{2\kappa} \{ \mathcal{L}_\xi (\sqrt{\sigma} n_\mu \delta u^\mu) - \delta (\sqrt{\sigma} n_\mu \mathcal{L}_\xi u^\mu) \} d^2x \quad (53)$$

Notice that, having not imposed the hypothesis of orthogonal intersections, the 2-form $\tau(L_H, X, \mathcal{L}_\xi g)$ does not vanish at the boundary B_t since n_μ in general is not surface-forming (instead, in presence of orthogonal boundaries, n_μ is surface-forming and hence δn_μ is proportional to n_μ so that $n_\mu \delta u^\mu =$

$-u^\mu \delta n_\mu = 0$). Hence formula (44) actually constitutes a generalization of previous definitions found in literature and based on Nöther theorem; see [12, 14, 32, 33].

The direct calculation of the expression (44) turns out to be rather cumbersome. Fortunately many results can be found elsewhere and we shall only reproduce the final results. First of all the bulk term in (44) has been already computed in [4, 5] even though it was originally calculated with respect to unbarred quantities. In terms of barred quantities it reads as follows:

$$\begin{aligned} \int_{\Sigma_t} \delta_X \tilde{\mathcal{E}}(L_H, \xi, \Sigma_t) &= \int_{\Sigma_t} \{ \delta N \mathcal{H} + \delta N^\alpha \mathcal{H}_\alpha + [h_{\alpha\beta}] \delta P^{\alpha\beta} - [P^{\alpha\beta}] \delta h_{\alpha\beta} \} d^3x \\ &\quad - \int_{B_t} d^2x \left\{ \bar{N} \delta(\sqrt{\sigma} \bar{\epsilon}) - \bar{N}^\alpha \delta(\sqrt{\sigma} \bar{j}_\alpha) + \frac{\bar{N} \sqrt{\sigma}}{2} \bar{s}^{\alpha\beta} \delta \sigma_{\alpha\beta} \right. \\ &\quad \left. + \frac{1}{\kappa} [\mathcal{L}_\xi(\sqrt{\sigma}) \delta(\theta) - \delta(\sqrt{\sigma}) (\mathcal{L}_\xi \theta)] \right\} \end{aligned} \quad (54)$$

where

$$\bar{\epsilon} = \frac{1}{\kappa} \bar{\mathcal{K}} \quad (55)$$

$$\bar{j}_\alpha = -\frac{2}{\sqrt{\gamma}} \sigma_{\alpha\mu} \bar{\Pi}^{\mu\nu} \bar{u}_\nu \quad (56)$$

$$\bar{s}^{\alpha\beta} = \frac{1}{\kappa} [(\bar{n}^\mu \bar{a}_\mu) \sigma^{\alpha\beta} - \bar{\mathcal{K}} \sigma^{\alpha\beta} + \bar{\mathcal{K}}^{\alpha\beta}] \quad (\bar{a}_\mu = \bar{u}^\nu \nabla_\nu \bar{u}_\mu) \quad (57)$$

are, respectively, the *surface quasilocal energy*, the *surface angular momentum* and the *surface stress tensor*, as viewed by the boosted observers. They are all tensors on the boundary surface B_t and, after integration on the surface, they can be used to describe the stress energy–momentum content of the gravitational field inside B_t ; see [1, 2, 4, 5] and reference quoted therein. Instead $[h_{\alpha\beta}]$ and $[P^{\alpha\beta}]$ in (54) are a shortcut for

$$[h_{\alpha\beta}] = \frac{2\kappa N}{\sqrt{h}} (2P_{\alpha\beta} - h_{\alpha\beta} P) + D_\alpha N_\beta + D_\beta N_\alpha \quad (58)$$

$$\begin{aligned} [P^{\alpha\beta}] &= -\frac{N\sqrt{h}}{2\kappa} (\mathcal{R}^{\alpha\beta} - \frac{1}{2} h^{\alpha\beta} \mathcal{R}) + \frac{\kappa N}{\sqrt{h}} h^{\alpha\beta} (P^{\mu\nu} P_{\mu\nu} - 1/2 P^2) \\ &\quad - \frac{2N\kappa}{\sqrt{h}} (P^{\alpha\mu} P_\mu^\beta - 1/2 P P^{\alpha\beta}) + \\ &\quad + \frac{\sqrt{h}}{2\kappa} (D^\alpha D^\beta N - h^{\alpha\beta} h^{\mu\nu} D_\mu D_\nu N) + \\ &\quad + D_\mu (P^{\alpha\beta} N^\mu) - D_\mu N^\alpha P^{\beta\mu} - D_\mu N^\beta P^{\alpha\mu} \end{aligned} \quad (59)$$

having denoted by $\mathcal{R}_{\alpha\beta}$ the Ricci tensor of the 3-metric $h_{\alpha\beta}$.

Let us now consider the surface terms into (44) to which we shall refer as $\delta_X H_{\text{surf}}(L_H, \xi, \Sigma_t)$. We shall carry out the calculation referring to the set of barred observers (the ones comoving with \mathcal{B}) instead of referring to unbarred

observers (at rest with respect to the Σ foliation). We shall also make repeated use of the property:

$$\int_{B_t} \mathcal{U} = \int_{B_t} \sqrt{\sigma} d^2 x \mathcal{U}^{[\alpha\beta]} n_\alpha u_\beta = \int_{B_t} \sqrt{\sigma} d^2 x \mathcal{U}^{[\alpha\beta]} \bar{n}_\alpha \bar{u}_\beta \quad (60)$$

which holds true for any 2-form $\mathcal{U} = 1/2 \sqrt{g} \mathcal{U}^{\alpha\beta} ds_{\alpha\beta}$.

The first term in the surface integral (44), after a bit of algebra (see [12, 13]) can be written as follows

$$\begin{aligned} \delta \left[\int_{B_t} \mathcal{U}_K(L_H, \xi) \right] &= \frac{1}{2\kappa} \int_{B_t} d^2 x \delta \left[2\sqrt{\sigma} \bar{u}^\mu \bar{\Theta}_\mu^\alpha \xi_\alpha \right] \\ &\quad + \frac{1}{2\kappa} \int_{B_t} d^2 x \delta \left[\sqrt{\sigma} \bar{u}_\alpha \mathcal{L}_\xi \bar{n}^\alpha \right] \end{aligned} \quad (61)$$

For the second term in $\delta_X H_{\text{surf}}(L_H, \xi, \Sigma_t)$ we obtain instead:

$$\begin{aligned} - \int_{B_t} i_\xi F(L_H, \xi) &= \frac{1}{2\kappa} \int_{B_t} \sqrt{\sigma} d^2 x \bar{N} \left[2\delta \bar{\Theta} + \bar{\Theta}^{\mu\nu} \delta \bar{\gamma}_{\mu\nu} \right] \\ &\quad - \frac{1}{2\kappa} \int_{B_t} d^2 x \mathcal{L}_\xi \left[\sqrt{\sigma} \bar{u}_\mu \delta \bar{n}^\mu \right] \end{aligned} \quad (62)$$

The third surface term $\int_{B_t} \tau$ has already been written down in (53), i.e.

$$- \int_{B_t} \tau(L_H, X, \mathcal{L}_\xi g) = \frac{1}{2\kappa} \int_{B_t} \left\{ -\mathcal{L}_\xi (\sqrt{\sigma} n_\mu \delta u^\mu) + \delta (\sqrt{\sigma} n_\mu \mathcal{L}_\xi u^\mu) \right\} d^2 x \quad (63)$$

The value of $\delta_X H_{\text{surf}}(L_H, \xi, \Sigma_t)$ results by the sum of (61), (62) and (63). In order to gain some simplification we shall make use of the relations (see [5]):

$$\bar{\Theta}_{\mu\nu} = \bar{K}_{\mu\nu} + (\bar{n}^\alpha \bar{a}_\alpha) \bar{u}_\mu \bar{u}_\nu + 2\sigma_{(\mu}^\alpha \bar{u}_{\nu)} \bar{K}_{\alpha\beta} \bar{n}^\beta \quad (64)$$

$$\delta \bar{\gamma}_{\mu\nu} = -(2/\bar{N}) \bar{u}_\mu \bar{u}_\nu \delta \bar{N} - (2/\bar{N}) \sigma_{\alpha(\mu} \bar{u}_{\nu)} \delta \bar{N}^\alpha + \sigma_{(\mu}^\alpha \sigma_{\nu)}^\beta \delta \sigma_{\alpha\beta} \quad (65)$$

$$n_\mu \delta u^\mu + \bar{u}_\mu \delta \bar{n}^\mu = -2\delta\theta \quad (66)$$

where

$$\sigma_\mu^\alpha \bar{K}_{\alpha\beta} \bar{n}^\beta = \sigma_\mu^\alpha (n^\beta K_{\alpha\beta} - \nabla_\alpha \theta) \quad (67)$$

After a little of algebra we reach the final result:

$$\begin{aligned} \delta_X H_{\text{surf}}(L_H, \xi, \Sigma_t) &= \int_{B_t} d^2 x \left\{ \bar{N} \delta (\sqrt{\sigma} \bar{\epsilon}) - \bar{N}^\alpha \delta (\sqrt{\sigma} \bar{j}_\alpha) + \frac{\bar{N} \sqrt{\sigma}}{2} \bar{s}^{\alpha\beta} \delta \sigma_{\alpha\beta} \right. \\ &\quad \left. + \frac{1}{\kappa} [\mathcal{L}_\xi (\sqrt{\sigma}) \delta (\theta) - \delta (\sqrt{\sigma}) (\mathcal{L}_\xi \theta)] \right\} \end{aligned} \quad (68)$$

Finally, the sum of the Hamiltonian bulk term (54) and the Hamiltonian surface term (68) gives the result

$$\delta_X H(L_H, \xi, \Sigma_t) = \int_{\Sigma_t} \left\{ \delta N \mathcal{H} + \delta N^\alpha \mathcal{H}_\alpha + [h_{\alpha\beta}] \delta P^{\alpha\beta} - [P^{\alpha\beta}] \delta h_{\alpha\beta} \right\} d^3 x \quad (69)$$

We stress that the boundary terms arising in the variation of the bulk term in (54) exactly cancel out the variation of the boundary term (68).

In order to explicitly write down the Hamilton equations (26) we still have to calculate the term

$$\int_{\Sigma_t} i_\xi e(L_H, X) = -\frac{1}{2\kappa} \int_{\Sigma_t} \sqrt{g} G^{\mu\nu} \delta g_{\mu\nu} \xi^\rho ds_\rho \quad (70)$$

Taking into account the relations

$$\begin{aligned} \delta g_{\mu\nu} &= -\frac{2u_\mu u_\nu}{N} \delta N - \frac{2h_{\alpha(\mu} u_{\nu)})}{N} \delta N^\alpha + h_{(\mu}^\alpha h_{\nu)}^\beta \delta h_{\alpha\beta} \\ \int_{\Sigma_t} \sqrt{g} f \xi^\rho ds_\rho &= \int_{\Sigma_t} \sqrt{h} f N d^3x \end{aligned}$$

we have

$$\begin{aligned} \int_{\Sigma_t} i_\xi e(L_H, X) &= \\ &= \frac{1}{2\kappa} \int_{\Sigma_t} \sqrt{h} d^3x [2G^{\mu\nu} u_\mu u_\nu \delta N + 2G^{\mu\nu} h_{\alpha\mu} u_\nu \delta N^\alpha - G^{\mu\nu} h_\mu^\alpha h_\nu^\beta \delta h_{\alpha\beta}] \end{aligned} \quad (71)$$

Taking into account (52), (69) and (71), the Hamilton equations (26), i.e.:

$$\delta_X H(L_H, \xi, \Sigma_t) = \int_{\Sigma_t} \tilde{\omega}(L_H, X, \mathcal{L}_\xi y) - \int_{\Sigma_t} i_\xi e(L_H, X) \quad (72)$$

read as follows:

$$-\frac{\sqrt{h}}{\kappa} G^{\mu\nu} u_\mu u_\nu = \mathcal{H} = 0 \quad (73)$$

$$-\frac{\sqrt{h}}{\kappa} G^{\mu\nu} h_{\alpha\mu} u_\nu = \mathcal{H}_\alpha = 0 \quad (74)$$

$$\mathcal{L}_\xi h_{\mu\nu} = [h_{\mu\nu}] \quad (75)$$

$$\frac{\sqrt{g}}{2\kappa} G^{\mu\nu} h_\mu^\alpha h_\nu^\beta = \mathcal{L}_\xi P^{\mu\nu} - [P^{\mu\nu}] = 0 \quad (76)$$

Notice that equations (73), (74) and (76) correspond to Einstein's equation in the (3+1) formalism while (75) is nothing but the definition of the momentum $P^{\mu\nu}$, i.e. the Legendre transformation.

Hence the prescription (44) originates the correct evolution equations in the phase space (h_{ab}, P^{ab}) of General Relativity and it does not give rise to additional boundary equations; see [9, 33].⁴ For this reason the definition (44)

⁴Notice, however, that Hamilton boundary equations would indeed arise if the correction term $\int_{B_t} \tau$ in formula (44) were missing.

can be truly considered as a genuine generalization of the Regge–Teitelboim Hamiltonian formulation [20] to arbitrary spacetime dimensions and to solutions with arbitrary asymptotic behavior (and not necessarily stationary).

Notice also the close formal analogy of expression (72) with the definition of Hamiltonian in Classical Mechanics. In Classical Mechanics one defines the generalized Hamiltonian through the usual rule $H(q, p)|_{p(q, \dot{q})} = p(q, \dot{q}) \dot{q} - L(q, \dot{q})$ so that

$$\delta H = \dot{q} \delta p - \dot{p} \delta q + (\dot{p} - \partial L / \partial q) \delta q \quad (77)$$

In trying to establish an analogy between (77) and definition (72) the term $\int_{\Sigma_t} \tilde{\omega}(L_H, \xi, \Sigma_t)$ in (72) can be seen as the $\dot{q} \delta p - \dot{p} \delta q$ term – see (52) – while the term $\int_{\Sigma_t} i_\xi e(L_H, X)$ clearly corresponds to Lagrange equations $(\dot{p} - \partial L / \partial q) \delta q$ – see (71).

The question arises whether a Hamiltonian H exists such that its variation δH equals the expression (69). The answer to the problem is given by:

$$\begin{aligned} H(L_H, \xi, \Sigma_t) - H_0(L_H, \xi, \Sigma_t) &= \int_{\Sigma_t} \{N\mathcal{H} + N^\alpha \mathcal{H}_\alpha\} d^3x \\ &+ \int_{B_t} \sqrt{\sigma} \{ \bar{N} \bar{\epsilon} - \bar{N}^\alpha \bar{j}_\alpha \} d^2x + \frac{1}{\kappa} \int_{B_t} \mathcal{L}_\xi(\sqrt{\sigma}) \theta d^2x \end{aligned} \quad (78)$$

where $H_0(L_H, \xi, \Sigma_t)$ turns out to be a constant of integration in the left hand side of (69). [It will play a fundamental role when dealing with the definition of energy since it will play the role of a reference term which defines the zero level for the energy itself]. Indeed, the variation of (78) turns out to be:

$$\begin{aligned} \delta_X H(L_H, \xi, \Sigma_t) &= \int_{\Sigma_t} \{ \delta N \mathcal{H} + \delta N^\alpha \mathcal{H}_\alpha + [h_{\alpha\beta}] \delta P^{\alpha\beta} - [P^{\alpha\beta}] \delta h_{\alpha\beta} \} d^3x \\ &- \int_{B_t} \left\{ \bar{\Pi}^{\mu\nu} \delta \bar{\gamma}_{\mu\nu} - \frac{1}{\kappa} \mathcal{L}_\xi(\theta \delta \sqrt{\sigma}) \right\} d^2x \end{aligned} \quad (79)$$

and it gives rise to expression (69) provided that, during the variation, the boundary metric $\bar{\gamma}_{\mu\nu}$ is kept fixed. [We also remark that it is indeed the Dirichlet condition $\delta \bar{\gamma}|_{\mathcal{B}} = 0$ together with the requirement that ξ be a boundary Killing vector field that ensures the definition of δH to be independent on the choice of a particular Cauchy hypersurface].

Moreover, expression (79) can be used to calculate the time rate of change of the Hamiltonian. To this end we have to replace the variations δ of the dynamical variables in (79) with their Lie dragging \mathcal{L}_ξ along the evolution vector field ξ . Assuming that Einstein’s equations are satisfied we obtain:

$$\mathcal{L}_\xi H(L_H, \xi, \Sigma_t) = - \int_{B_t} \left\{ \bar{\Pi}^{\mu\nu} \mathcal{L}_\xi \bar{\gamma}_{\mu\nu} - \frac{1}{\kappa} \mathcal{L}_\xi(\theta \mathcal{L}_\xi \sqrt{\sigma}) \right\} d^2x \quad (80)$$

which is vanishing if $\mathcal{L}_\xi \bar{\gamma}_{\mu\nu} = 0$, i.e. if the vector field ξ is a boundary Killing vector field for the metric $\bar{\gamma}$ or, in other words, if the gravitational system is isolated from the outside; see [1, 2, 4].

3.2 Energy in General Relativity

Let us now consider the definition of energy. It could be obtained simply by evaluating (78) on-shell. In order to better understand the role played by the reference term H_0 and its relations with boundary conditions let us consider again the on-shell variation of the Hamiltonian. We specialize definition (28) for the Hilbert Lagrangian L_H so that $\delta_X E(g) = \delta_X H_{\text{surf}}(L_H, \xi, \Sigma_t)$ and from (68) one has:

$$\begin{aligned} \delta_X E(g) = \int_{B_t} d^2x \left\{ \bar{N} \delta(\sqrt{\sigma} \bar{\epsilon}) - \bar{N}^\alpha \delta(\sqrt{\sigma} \bar{j}_\alpha) + \frac{\bar{N} \sqrt{\sigma}}{2} \bar{s}^{\alpha\beta} \delta \sigma_{\alpha\beta} \right. \\ \left. + \frac{1}{\kappa} [\mathcal{L}_\xi(\sqrt{\sigma}) \delta(\theta) - \delta(\sqrt{\sigma}) (\mathcal{L}_\xi \theta)] \right\} \end{aligned} \quad (81)$$

Notice that $\delta_X E(g)$ is a pure surface integral on B_t . Nevertheless we stress that definition (81) *is not* a quasilocal energy relative to the surface B_t . Indeed, let us now consider a hypersurface Σ_t with an external boundary B_t and an internal boundary C_t , i.e. $\partial \Sigma_t = B_t \cup C_t$. Taking expression (52) into account, equation (29) now reads as:

$$\begin{aligned} \delta_X E(g)|_{\partial \Sigma_t} &= \delta_X E(g)|_{B_t} - \delta_X E(g)|_{C_t} \\ &= \int_{\Sigma_t} [(\mathcal{L}_\xi h_{\mu\nu}) \delta P^{\mu\nu} - (\mathcal{L}_\xi P^{\mu\nu}) \delta h_{\mu\nu}] d^3x \end{aligned} \quad (82)$$

For stationary solutions the right hand side vanishes together with the correction term $\tau(L_H, X, \xi)$, so that $\delta_X E$ has the same numerical value when computed on different surfaces belonging to the same homology class. In particular, if spatial infinity is homologous to the surface B_t , expression (81) corresponds to the total energy or, better, it corresponds to the total mass. Applications to a number of specific solutions, including e.g. Kerr–Newman, BTZ, Taub–bolt solutions, can be found in [14, 30, 31]. In all the examples so far analysed it has been shown that expression (81) (with $\tau = 0$) always gives rise to the expected value, namely, the total mass of the solution.

As previously remarked an expression of the quasilocal energy can be instead obtained only when boundary conditions are imposed on the surface B_t . We define the *quasilocal internal energy* $E(g, B_t)$ contained in the surface B_t to be the value obtained from (81) by imposing Dirichlet metric boundary conditions. Namely, we consider a one parameter family of solutions of field equations which admit the same boundary metric $\bar{\gamma}_{\mu\nu}$ and we consider a vector field X tangent to this family, i.e.

$$\delta \bar{\gamma}_{\mu\nu}|_{\mathcal{B}} = 0 \quad (83)$$

Since $\delta \bar{\gamma}_{\mu\nu}|_{\mathcal{B}} = 0$ implies $\delta \bar{N}|_{\mathcal{B}} = 0$, $\delta \bar{N}^\alpha|_{\mathcal{B}} = 0$ and $\delta \sigma_{\mu\nu}|_{\mathcal{B}} = 0$, formula (81) can be explicitly integrated:

$$E(g, B_t) - E_0(g_0, B_t) = \int_{B_t} d^2x \left\{ \sqrt{\sigma} (\bar{N} \bar{\epsilon} - \bar{N}^\alpha \bar{j}_\alpha) + \frac{1}{\kappa} \mathcal{L}_\xi(\sqrt{\sigma}) \theta \right\} \quad (84)$$

We stress that we are integrating along a family of solutions satisfying the boundary condition (83) and thence the term $E_0(g_0, B_t)$ corresponds to the quasilocal energy relative to a solution g_0 inside the family itself, which then becomes the reference point (or zero level) for the energy. Owing to the properties $\bar{N} = \bar{N}_0|_{\mathcal{B}}$, $\bar{N}^\alpha = \bar{N}_0^\alpha|_{\mathcal{B}}$ and $\sigma_{\mu\nu} = \sigma_{0\mu\nu}|_{\mathcal{B}}$, expression (84) can be equivalently rewritten as:

$$E(g, B_t) = \int_{B_t} d^2x \left\{ \sqrt{\sigma} [\bar{N}(\bar{\epsilon} - \bar{\epsilon}_0) - \bar{N}^\alpha(\bar{j}_\alpha - \bar{j}_{0\alpha})] + \frac{1}{\kappa} \mathcal{L}_\xi(\sqrt{\sigma})(\theta - \theta_0) \right\} \quad (85)$$

where the subscript 0 clearly refers to the background solution g_0 . In other words, the background terms can be considered as resulting from the embedding of $(B_t, \sigma_{\mu\nu})$ into a reference spacetime and by requiring that the surface evolution is the same in M and in the reference spacetime, i.e. by requiring that the boundary lapses \bar{N} and \bar{N}_0 coincide, the boundary shifts \bar{N}^α and \bar{N}_0^α are equal as well as the time rates of change $\mathcal{L}_\xi(\sqrt{\sigma})$ of the metric σ in the two spacetimes are the same; see [4]. Roughly speaking, expression (85) defines the quasilocal energy of the solution g *relative* to a solution g_0 satisfying the same Dirichlet conditions on the boundary. From a symplectic viewpoint we are selecting the six components of the boundary metric $\bar{\gamma}_{\mu\nu}$ as the control parameters of the gravitational system, while the response variables are the relative quasilocal surface energy $\bar{\epsilon} - \bar{\epsilon}_0$, the relative quasilocal surface angular momentum $\bar{j}_\alpha - \bar{j}_{0\alpha}$ and the relative velocity parameter $\theta - \theta_0$.

We stress that formula (85) differs from the expression of quasilocal internal energy as given in [4] only for the addition of the term $\mathcal{L}_\xi(\sqrt{\sigma})(\theta - \theta_0)$. [Indeed, in [4] the term $\theta \mathcal{L}_\xi(\sqrt{\sigma})$ was considered as a $p\dot{q}$ boundary term and thereby it was not included in the definition itself of the Hamiltonian]. Hence formula (85) perfectly agrees with the expression given in those papers when the control parameter $\sqrt{\sigma}$ is constant in time, i.e. $\mathcal{L}_\xi(\sqrt{\sigma}) = 0$ (or when the intersection parameter θ equals the intersection parameter θ_0 in the reference spacetime). Otherwise expression (85) contains an additional contribution to the energy coming from the accretion in time of the surface. However, we stress that the quasilocal energy contained in B_t depends not only on B_t itself but also on its history, geometrically described by the world tube \mathcal{B} . Notice that we can always make use of the condition $\mathcal{L}_\xi(\sqrt{\sigma}) = 0$ to assign a world tube \mathcal{B} to B_t or, equivalently, to assign an evolution vector field ξ to B_t ; see [9]. In practice, however, one always assumes that the evolution vector field is a Killing vector field on the boundary. Indeed, this requirement guarantees the conservaton of energy in time, where the time is defined by the flow of the vector field. Provided that these remarks are taken into account, the term $\mathcal{L}_\xi(\sqrt{\sigma})(\theta - \theta_0)$ can be set equal to zero so that formula (85) agrees with the one calculated in [4].

Another expression for the energy of the gravitational field in the case of a non-orthogonal foliation of spacetime was previously calculated in [8]. The calculation there developed was carried out with respect to the foliation of spacetime as a whole rather than with respect to the foliation of the boundary, as instead explicitly done in this Section.

The expression there presented, rewritten according to our notation, reads as follows:

$$\begin{aligned} E_{HH}(g) - E_{HH}(g_0) &= \frac{1}{\kappa} \int_{B_t} d^2x \sqrt{\sigma} \left[N\mathcal{K} - n_\mu N_\nu \left(\frac{2\kappa}{\sqrt{h}} P^{\mu\nu} \right) \right] \\ &\quad + \frac{1}{\kappa} \int_{B_t} d^2x \sqrt{\sigma} N \bar{u}^\mu \nabla_\mu \theta \end{aligned}$$

The background solution g_0 was then chosen in [8] by requiring that g and g_0 induce the same 3-metric on the boundary \mathcal{B} and the same intersection angle θ on B_t . Hence:

$$\begin{aligned} E_{HH}(g) &= \frac{1}{\kappa} \int_{B_t} d^2x \sqrt{\sigma} N [\mathcal{K} - \mathcal{K}_0] \\ &\quad - 2 \int_{B_t} d^2x \sqrt{\sigma} \left[n_\mu N_\nu \left(\frac{P^{\mu\nu}}{\sqrt{h}} \right) - n_{0\mu} N_{0\nu} \left(\frac{P_0^{\mu\nu}}{\sqrt{h_0}} \right) \right] \end{aligned} \quad (86)$$

Taking into account the relation

$$N_\alpha = h_\alpha^\nu N_\nu = (\sigma_\alpha^\nu + n^\nu n_\alpha) N_\nu \stackrel{(42)}{=} \sigma_\alpha^\nu N_\nu + N v n_\alpha \quad (87)$$

together with the splitting (see [5])

$$K_{\mu\nu} = l_{\mu\nu} + (u \cdot b) n_\mu n_\nu + 2 \sigma_{(\mu}^\alpha n_{\nu)} K_{\alpha\beta} n^\beta \quad (b^\mu = n^\nu \nabla_\nu n^\mu) \quad (88)$$

(where $l_{\mu\nu} = -\sigma_\mu^\alpha \sigma_\nu^\beta \nabla_\alpha u_\beta$ and $b^\mu = n^\nu \nabla_\nu n^\mu$), expression (86) can be further rewritten as:

$$E_{HH}(g) = \int_{B_t} d^2x \sqrt{\sigma} \left\{ \frac{N}{\kappa} [(\mathcal{K} - \mathcal{K}_0) + v(l - l_0)] - N^\beta \sigma_\beta^\alpha (j_\alpha - j_{0\alpha}) \right\} \quad (89)$$

(with $l = \sigma^{\mu\nu} l_{\mu\nu}$ and $l_0 = \sigma_0^{\mu\nu} l_{0\mu\nu} = \sigma^{\mu\nu} l_{0\mu\nu}$).

In order to establish a correspondence between expression (85) and the definition (89), we first have to rewrite (85) in terms of “unboosted” quantities. To reach this goal we shall make use of the boost relations (see again [5]):

$$\bar{N} = N/\gamma \quad (90)$$

$$\bar{\mathcal{K}} = \gamma \mathcal{K} + \gamma v l \quad (91)$$

$$\bar{j}_\alpha = j_\alpha - \frac{1}{\kappa} \nabla_\alpha \theta \quad (92)$$

(where $j_\alpha = \frac{1}{\kappa} \sigma_\alpha^\beta K_{\beta\mu} n^\mu = -\frac{2}{\sqrt{h}} \sigma_{\alpha\mu} P^{\mu\nu} n_\nu$) together with the relation

$$\mathcal{L}_\xi(\sqrt{\sigma}) = \sqrt{\sigma} [-N l - N v \mathcal{K} + \mathfrak{D}_\alpha(\sigma_\beta^\alpha N^\beta)] \quad (93)$$

where \mathfrak{D}_α denotes the (metric) covariant derivative on the surface. After a straightforward calculation it is easy to show that expression (85) exactly agrees

with (89) provided the same matching conditions are imposed between g and the background solution g_0 .

For the sake of completeness we recall that a formula for the energy of the gravitational field in the case of a non-orthogonal foliation of spacetime can be also found in [9]. It was there obtained through a new method of variation of the gravitational Lagrangian and through a Legendre transformation technique. We end this section by noticing that expressions (81) and (82) together reproduce the “homogenous” formula (80) of Kijowski’s paper [9] which is at the basis of his formalism.

In this way a (conditioned) correspondence among the various definitions of energy [4, 8], [1, 5] and [9] (based, respectively, on the canonical analysis, on Hamilton–Jacobi analysis and on symplectic analysis) and the Nöther approach is established.

3.3 Entropy

In the previous section we have shown that formula (81) leads to the definition of quasilocal internal energy provided that Dirichlet boundary conditions are imposed on the metric field. Formula (81) can be also used in order to describe the first law of black hole thermodynamics. Indeed, the direct evaluation of (81) on the horizon of a black hole solution and the boundary conditions dictated by the geometry of the black hole horizon lead to the formulation of the first law of black hole thermodynamics.

Indeed, let us specialize expression (81) to an axisymmetric stationary black hole solution $g_{\mu\nu}$ of Einstein (vacuum) field equations. On each leaf Σ_t let us choose spatial coordinates x^i co-rotating with the horizon. Hence the vector field ξ coincides with the null vector field generating the horizon. We recall that, with this choice of coordinates, the metric near the horizon satisfies the regularity condition:

$$\bar{n}^\mu \partial_\mu N = \kappa_H \quad (\kappa_H = \text{surface gravity}) \quad (94)$$

which ensues from the necessity of removing the conical singularity in the complex metric obtained from the Lorentzian metric by analytically continuing time to imaginary values (see [2], [36], [37]). We claim that formula (81) already encloses all the information about the first law, in the same way it was originally formulated in [2], but now generalized to non-orthogonal intersections (see also [40]). To realize this, first of all notice that the integrand in the right hand side of formula (81) is a closed 2-form as ξ is a Killing vector field of the solution g (and hence the right hand side of (82) is vanishing). By Stokes’ theorem, we have that $\delta_X E(g)|_{B_t} = \delta_X E(g)|_{C_t}$ if C_t is any 2-surface homologous to B_t , i.e. if $B_t - C_t$ is the boundary of a 3-dimensional region Σ_t . We can choose, for example, $C_t = \Delta$, where Δ denotes the (cross section of the) black hole horizon. In that case the lapse and the shift are vanishing on the horizon and, taking the boundary condition (94) into account the only term surviving in the right hand

side of (81) is (see [40]):

$$\delta_X E(g)|_\Delta = \frac{\kappa_H}{\kappa} \delta \left(\int_H d^2 x \sqrt{\sigma} \right) = \frac{\kappa_H}{\kappa} \delta A_H \quad (95)$$

where A_H denotes the area of the horizon. In order to attribute a thermodynamical meaning to the dynamics of black holes, in accordance with [2] we define the following intensive variables: the *inverse temperature* β of the surface, the *proper angular velocity* ω and the *surface pressure* p^{ij} . They are defined, respectively, as follows:

$$\begin{aligned} \beta &= \frac{2\pi}{\kappa_H} \bar{N} \\ \beta\omega &= \frac{2\pi}{\kappa_H} \bar{N}^\phi \\ \beta \frac{\sqrt{\sigma}}{2} p^{ij} &= \frac{2\pi}{\kappa_H} \bar{N} \frac{\sqrt{\sigma}}{2} \bar{s}^{ij} \end{aligned} \quad (96)$$

As one can read off from (81) they correspond to $2\pi/\kappa_H$ times the quantities conjugated, respectively, to $\sqrt{\sigma}\bar{\epsilon}$, $\sqrt{\sigma}\bar{j}_\alpha$ and $\sigma_{\alpha\beta}$, i. e. conjugated to the extensive variables (however, we stress that the physical interpretation of quasilocal quantities in terms of thermodynamical variables comes from a path integral formulation of gravity – see [35, 2] – and it is then out of the scopes of our discussion). Equating the left hand sides of equations (81) with (95) and taking the property of stationarity into account we then obtain the first law of black hole thermodynamics for spatially bounded system as formulated in [2] but generalized now to non-orthogonal intersections:

$$\int_{B_t} d^2 x \left\{ \beta \delta(\bar{\epsilon} \sqrt{\sigma}) - \beta \omega \delta(\bar{j} \sqrt{\sigma}) + \beta \frac{\sqrt{\sigma}}{2} p^{ij} \delta \sigma_{ij} \right\} = \frac{\delta A_H}{4} \quad (97)$$

(in geometric units $G = c = 1$ where $\kappa = 8\pi$). As already remarked in [2], the equation (97) resembles the thermodynamical law $T\delta S = \delta E - \omega\delta J + p\delta V$ of thermodynamical systems provided we define $\beta = 1/T$ and $S = A_H/4$ (and it reduces exactly to it in static, spherically symmetric systems when β , ω and p^{ij} can be pulled out of the integral).

We point out that the equality in formula (97) is a trivial consequence of homological properties, namely of Stokes' theorem. For this reason the formula can be generalized to solutions other than black hole ones and, of course, it does not rely on horizons. Additional boundaries (i.e. additional singularities other than a single black hole one) are responsible of additional contributions to the entropy. In this way, the homological nature of entropy becomes evident; see [31], [38], [39].

We also stress that formula (81) is also well-suited to describe the thermodynamics of *isolated horizons*; see [41]. Indeed evaluating (81) on a isolated horizon and taking into account the boundary conditions dictated by the definition itself of isolated horizons, the first principle is recovered. We shall analyse this matter in the forthcoming paper [42].

4 Acknowledgments

We are grateful to G. Allemandi, L. Fatibene and M. Ferraris of the University of Torino for useful discussion on the subject. This work has been partially supported by the University of Torino (Italy).

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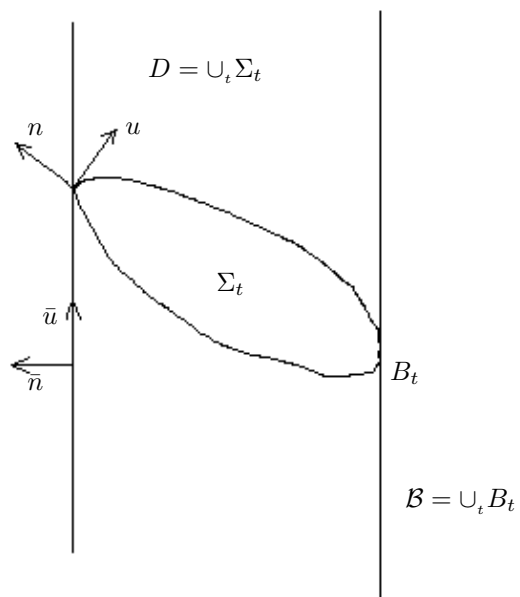


Figure 1: non-orthogonal foliation of a spacetime region D